Value-at-Risk Estimation with Multivariate Garch Models

UDC: 005.334:336.76 ; 330.43

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XII International Symposium SymOrg 2010, 09 - 12 June 2010, Zlatibor, Serbia

Value-at-Risk (VaR) method has been accepted globally by both risk managers and regulators as a tool to identify and control exposure to financial market risk. Basel II regulation employs VaR methodology for capital requirements calculations for the market risks to which commercial banks are exposed. The goal of this paper is to implement the multivariate GARCH (mGARCH) methodology as the internal VaR model for market risk measurement in Serbian commercial banks. Assuming Normal and Student-t distribution of the returns the parameters for orthogonal mGARCH and CCC-mGARCH VaR models are estimated for each of 250 consecutive days, for the hypothetical trading portfolio, by employing maximum likelihood method. The level of capital requirements are calculated for corresponding VaR methods and validation is done by applying Basel II and Kupiec test.

1. Introduction

The practice of risk management is evolving rapidly, especially with the implementation of the latest regulatory standards of the Basel Committee on Banking Supervision, known as Basel II. Its main goal is to implement a set of standards for risk measurement and management that would lead to adequate assessments of the levels of capital requirements that banks and other financial institutions need to keep as a cushion against various types of risk.

The Basel II provides an incentive scheme for institutions to develop their own internal risk management models. More precisely, banks have the option to use internal risk measurement models, IMA approach, to determine their capital charge. The rationale is that banks are in a position to produce more accurate measures of their individual risk exposure with respect to a general simplistic scheme proposed by regulators.

Besides credit activities, proprietary trading, which is a source of market risk, has become one of the emphasized activities of banks. In an unstable economic environment where asset prices are volatile, this activity brings a significant amount of market risk exposure. Therefore, there is a need for adequate risk management models and tools that will better control and mitigate those risks.

According to [4], market risk is defined as the risk of losses in on- and off-balance-sheet positions arising from movements in market risk factors. The main sources of market risk are the risks related to interest rate related instruments, equities, foreign exchange, derivatives and commodity related instruments. Moreover, market risk management should be conducted as a regular activity of a bank, under the jurisdiction of the risk management unit and independently from the trading sector [3].

Current regulatory standards in Serbia require the banks to measure and report market risk in its trading book, as well as to hold capital to cover potential losses. National Bank of Serbia provides a set of predefined regulatory standards in the form of tabulated reports, which banks are required to submit on a monthly basis. This supervisory framework is relatively conservative, requiring a capital charge of 12 percent of total risk-weighted assets [17]. The procedures for calculating these capital charges are in good part based on the Standardized Approach of Basel II. Moreover, Serbian banks are required to fully adopt Basel II standards by 2011, which would stimulate banks to use more advanced approaches in the future in order to reduce their capital charges.

The choice of a market risk measurement model is far from a "one-size-fits-all" procedure. In the same vain, institutions adopting internal models must ensure that their models are valid. The contemporary market risk management practice involves various "Value-at-Risk" (VaR) methods to be applied and implemented with the full compliance with Basel II standards. The Value-at-Risk has gained recognition as the primary tool for market risk measurement in banks and there is a widespread agreement on the use of it. However, there is very little consensus on the preferred method for calculating VaR. The difficulty in obtaining reliable VaR estimates stems from the fact that all extant methods involve some tradeoffs, assumptions and simplifications. Thus, determining what is the best methodology for VaR estimation becomes an empirical question and a question of implementation.

This paper considers several possible multivariate GARCH methodologies for advanced VaR estimation in trading portfolio of a bank. Those methodologies will be theoretically evaluated and applied on an example of a portfolio of assets traded in the Serbian capital market which may be hypothetically held by a particular Serbian bank.

Additionally, each of following VaR methods has been coded in MATLAB enabling estimation to become automated and capable of handling VaR assessment for a great number of financial instruments in trading portfolio.

2. The value-at-risk framework

The main strength of VaR is that it summarizes the exposure to market risk and it provides an aggregate view of overall portfolio's risk. The VaR estimation process, however, involves selecting two important parameters: the holding period and the confidence level. By definition, VaR measures the maximum loss in a portfolio value due to adverse market movements over a specified period of time with a given level of confidence. For instance, we estimated VaR = 50,000 RSD for a holding period of 1 day and a confidence level of cl = 99%. This indicates that the portfolio loss is not likely to exceed 50,000 RSD during the next trading day, with a 99% probability. For most applications, the recommendation is to choose a confidence level such as 95 to 99 percent and holding period 1 or 10 days[1].

The VaR estimate represents a point of the distribution of portfolio profit and loss such that if we assume some probability and thinking about 1-day losses, we can say informally that VaR is the minimum amount the bank will loose on a trading portfolio on a bad day, or the maximum it can expect loose on a good day. Generally, VaR models have to deal with four mathematical components:

The VaR modeling technique is based on two main approaches. *Univariate VaR modeling* is the way of estimating VaR by holding or assuming a single asset in a trading portfolio of a bank, or alternatively of calculating and using single portfolio return series based on weighted sum of portfolio component returns. The idea is to base VaR estimation upon single return series which must be able to capture behavior of all risk factor components. On the other hand, *Multivariate VaR modeling* assumes to take into consideration the returns time series of all portfolio constituents. In general portfolios have multiple n assets, thus in order to base VaR calculation upon risk impact of each portfolio component one has to capture n time series effect into analysis[2].

The VaR estimation in this paper is done according to *multivariate* methodology. In order to capture the probability density function $f_p(\cdot)$ established by the hypothetical portfolio returns or portfolio assets time series the first step in modeling VaR is to calculate returns on each portfolio asset component as follows:

$$r_{i,t} = \left(P_{i,t} - P_{i,t-1}\right) / P_{i,t-1} \tag{1}$$

where $r_{i,t}$ denotes the arithmetic return on asset i at time t, $P_{i,t}$ is the price of the asset i at time t and $P_{i,t-1}$ is the price of the asset i at time $t-1$. Consequently, the hypothetical portfolio return at time t, given N assets, is defined as the weighted sum of portfolio constituent returns:

$$r_{p,t} = \sum_{i=1}^{N} w_{i,t} r_{i,t} \tag{2}$$

It is important to note that the VaR measure represents one-day-ahead forecast of loss. In order to check the validity of VaR model i.e. potential breach[3], VaR at time t is estimated with respect to the information set ψ_t and then compared with the corresponding amount of profit/loss incurred at $t+1$. Consider a portfolio whose price at $t+1$ is labeled by p_{t+1}. The variation observed from a day to day is given as $\Delta p_{t+1} = p_{t+1} - p_t$. Note that if Δp_{t+1} is positive we have a profit, while a negative value indicates a loss. The $VaR_{1-\alpha}$ is defined in monetary units, so that the variation Δp_{t+1} observed for a portfolio will only be less than VaR with a probability of α where (1- α) represents confidence level:

$$Pr\left[\Delta p_{t+1} \leq -VaR_{t,1-\alpha}\right] = \alpha \tag{3}$$

The choice of a suitable portfolio distribution for modeling of asset and portfolio returns is an essential step in estimating VaR. At first glance, the general shape of a majority of empirical returns distributions, especially in the cases of well diversified portfolios, has indicated that the Normal distribution would be a natural assumption. Thus, when we assume that returns follow a Normal distribution with mean μ_t and volatility σ_t Equation (3) can be changed to:

$$Pr\left[\frac{\Delta p_{t+1} / p_t - \mu_t}{\sigma_t} \leq \frac{-VaR_{t,1-\alpha} / p_t - \mu_t}{\sigma_t}\right] = \alpha \tag{4}$$

[1] **The choice of the holding period depends on the characteristics of the portfolio that is held by a bank and the use of VaR. For example, if the positions changes quickly, the short horizon will be appropriate. If the purpose is to provide an accurate benchmark measure of downside risk, the horizon should be also ideally less than the average period for major portfolio rebalancing.**

[2] **As a result in such applications one needs to forecast the covariance matrix of all the assets in portfolio. Consequently, estimation of VaR will have to deal with n time series and to form corresponding covariance matrix.**

[3] **Breach is captured each time when the tomorrow's realized loss incurred in portfolio is greater than today's estimate of VaR**

This shows that the right inequality side of equation (4) is the quantile of the standard Normal distribution expressed as $Z_\alpha = -Z_{1-\alpha}$. Thus, the VaR amount of money by univariate estimation methodology is calculated:

$$VaR_{t,1-\alpha} = -(\mu_t - Z_{1-\alpha} \cdot \sigma_t) \cdot p_t \qquad (5)$$

If the VaR is defined as percentage relative to portfolio value as $\%VaR_{t,1-\alpha} = VaR_{t,1-\alpha} / p_t$ then we have:

$$Pr[r_{t+1} \leq -\%VaR_{t,\alpha}] = \alpha \qquad (6)$$

In practice, instead of working with (5) which denotes so-called "absolute" VaR, we can assume that μ_t=0 and use the "relative" VaR defined as:

$$VaR_{t,\alpha} = Z_{1-\alpha} \cdot \sigma_t \cdot p_t \qquad (7)$$

The "relative" VaR does not require that we know the first moment of Normal distribution μ. Furthermore, as we are dealing with a shorter time periods, one day return frequency, the difference between absolute and relative VaR will be fairly small.

In the multivariate framework it is required to shift from an individual asset/or portfolio position to a portfolio case where multiple positions affect VaR estimation. Not only the volatilities of individual returns, but also their covariances need to be taken into account. Thus, estimating VaR of the portfolio of asset positions which are sensitive to several different market risk factors therefore requires an additional input i.e. covariance matrix among market factor returns. Thus the equation (7) becomes:

$$VaR_{t,\alpha} = Z_{1-\alpha} \cdot \sqrt{\mathbf{w}^{\mathbf{T}}\mathbf{V}\mathbf{w}} \cdot P_t \qquad (8)$$

where the vector of asset weights in portfolio is $\mathbf{w} = (w_1, ..., w_n)^T$, P_t denotes the current portfolio value and $\mathbf{V}$ represents the corresponding covariance matrix of the portfolio's assets returns.

Another alternative way of calculating VaR using multivariate framework employs vector of position values $\mathbf{P} = [P_1, P_2,\ldots,P_n]^T$ of portfolio constituents instead of vector of corresponding assets weights **w**: Thus we have multivariate form of a relative VaR denoted by the equation:

$$VaR_{t,\alpha} = Z_{1-\alpha} \cdot \sqrt{\mathbf{P}^{\mathbf{T}}\mathbf{V}\mathbf{P}} \cdot \qquad (9)$$

Empirical evidence has shown that the assumption of normally distributed returns is usually not justified. Unlike the predicted 'normal' behavior, observed distributions of asset returns sometimes show a significant degree of skewness and high kurtosis. The property of having more weight in the tails than would be expected under the normal distribution has significantly large impact on VaR estimation. When the data are 'heavy tailed', the true probability of a large negative return is greater than the one predicted by the normal distribution. This implies that VaR calculated using the assumption of normally distributed returns can significantly understate the risk of a high loss, especially at high confidence levels. Consequently in this paper, we also considered one of the most commonly used alternatives which take into account non-normality of asset returns, namely we also applied the Student's *t*-distribution as the underlying assumption of assets returns behavior.

The expression for VaR assuming Student's *t*-distribution can be easily derived by altering α-quantile of Normal distribution defined as $Z_{1-\alpha}$, with corresponding α-quantile of the Student's *t*-distribution denoted as $\chi_{\alpha,\nu}$,with „degrees of freedom", into corresponding VaR equations previously considered. The Student's *t*-distribution is closely related to the Normal, but generally it has fatter tails depending on the value of an „degrees of freedom" parameter. By adopting the "degrees of freedom" the level of kurtosis can be modeled to match the kurtosis present in the observed time series. As a result, univariate *t*-VaR equation becomes:

$$VaR_{t,\alpha} = \chi_{1-\alpha,\nu} \cdot \sqrt{(\nu-2)/\nu} \cdot \sigma_{p,t} \cdot P_t \qquad (10)$$

By inspection, it can be inferred that the *t*-VaR formula includes the additional multiplier term $\sqrt{(\nu-2)/\nu}$ which moderates the effect of the standard deviation term of the previous VaR equation.

In multivariate framework we have to assure that each of assets return series is modeled separately according to assumed Student's *t*-distribution. As a result, there is a need to accommodate variance covariance matrix with additional multiplier term for each asset in portfolio. The number of different multipliers applied equals the number of different portfolio assets time series. Thus, the idea is to affect each component of variance covariance matrix:

$$VaR_{t,\alpha} = \sqrt{\tilde{\mathbf{P}}^{\mathbf{T}}\mathbf{V}\tilde{\mathbf{P}}} \qquad (11)$$

The accommodation is done through adjusted vector of position values $\tilde{\mathbf{P}}$ where additional multiplier term which includes the effect of taking into account estimated degrees of freedom separately for each corresponding asset in portfolio:

$$\tilde{\mathbf{P}}^{\mathbf{T}} = \lfloor \chi_{'1-\alpha,\nu_1} \cdot \sqrt{(\nu_1-2)/\nu_1} \cdot P_1 \quad \cdots \quad \chi_{'1-\alpha,\nu_n} \cdot \sqrt{(\nu_n-2)/\nu_n} \cdot P_n \rfloor \qquad (12)$$

The quantile terms for denoted by $\chi_{'1-\alpha,\nu_1}, \chi_{'1-\alpha,\nu_2}, \ldots, \chi_{'1-\alpha,\nu_n}$ now depend on the chosen confidence level, α, as well as on the number of degrees of freedom of portfolio component. Since the Student's *t*-distribution converges to the Normal distribution as gets large, we can regard the Student's *t* VaR with a finite ν as a generalization of the normal VaR. As ν gets large, approaches its normal equivalent $Z_{1-\alpha,\nu}$, and the additional multiplier term approaches one.

The analytical VaR models discussed above are the simplest ones among VaR framework. These models consider volatility and correlations as constant parameters over time and assume the relevant portfolio assets returns to be characterized by a stable distribution over time. This assumption is clearly in contrast with the empirical evidence, which shows that volatility and correlations vary over time.

3. The mulivariate garch modeling of value-at-risk

The phenomenon which is often referred to as "volatility clustering" indicates that asset returns often experience periods of low and high volatility. The Figure 1. shows that the assumption that volatility is constant over time, which is a hypothesis for unconditional VaR models[4], may be misleading.

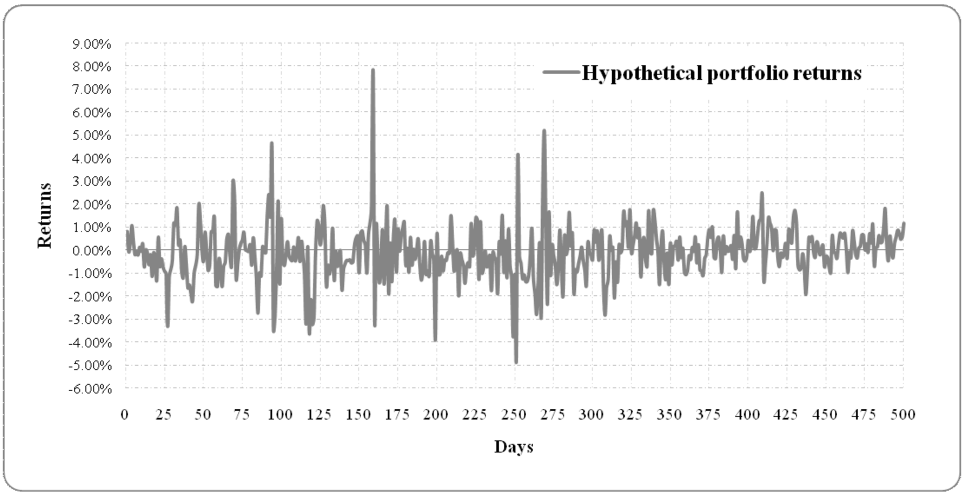

Figure 1: *Periods of high and low volatility*

The volatility clustering effect can be explicitly handled by GARCH i.e. generalized autoregressive conditional heteroskedasticity models[5]. The GARCH models are able to capture the sophisticated effects in the volatility behavior. Beside already mentioned volatility clustering effect GARCH implies another important property referred to as mean reversion. In this context, mean reversion means that in the absence of innovation variance tends towards some long-run equilibrium level.

Univariate GARCH (1,1). Assuming that the residuals are conditionally normally distributed, a GARCH (p,q) model can be specified as follows:

$$r_t = \mu_t + \varepsilon_t \tag{13}$$

$$\varepsilon_t \sim N\left(0, \sigma_t^2\right) \tag{14}$$

$$\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{15}$$

where $\alpha_1 < 1$, $\beta_1 < 1$ and $\alpha_1 + \beta_1 < 1$. The Equation (13) indicates that the return at time *t*, r_t, is composed of deterministic part μ_t, and a random one ε_t. The ε_t ÿstands for the 'innovation' at time *t* and it represents a sequence random shocks with mean zero and variance shown in Equation (14). As a result conditional variance at time *t* represented by Equation (15) is specified as a function of three factors: constant ω the variance estimated in the previous period $\sigma^2_{\tau=1}$ and the squared $\varepsilon^2_{\tau=1}$ innovation at *t-1*. Thus the conditional variance estimate in a certain period is a weighted mean of long-term variance, the expected variance for the previous period, and a shock for the last period. The estimate of the unconditional, i.e. "theoretical" long-term, value of variance is implied by the model. If such a value exists, it will represent the unconditional expected value such that $\sigma^2 = E\left[\varepsilon_{t-1}^2\right] = E\left[\sigma_t^2\right] = E\left[\sigma_{t-1}^2\right]$ and the following will be obtained:

$$\sigma^2 = \frac{\omega}{1 - \alpha_1 + \beta_1} \tag{16}$$

The majority of applications of the GARCH models are based upon the GARCH (1, 1) which is the most widely used GARCH model in practice. The main reason for this is that, most often, GARCH (1, 1) fits the data acceptably well.

[4] the key difference between unconditional and conditional models is related to the fact that the former gives a constant as an estimates while the latter needs a specification model and regression technique dependent on time for an estimate to be done.

[5] *Heteroskedasticity* means time-changing variance and it is in contrast to the constant variance notion. *Conditional* indicates that the predictions obtained are based upon the information available in the previous period, so for example, the current level of volatility reflects the current level of uncertainty generated by past shocks. *Autoregressive* refers to the method used to model conditional heteroskedasticity which is based on variance self regression. Finally, *generalized* refers to a particular type of model which was introduced as a generalization of the first autoregressive conditional variance (ARCH) model. Thus, the autoregressive conditional heteroskedasticity models therefore allow predicting future volatility by using a regression based upon the past values of volatility estimates.

Multivariate GARCH (mGARCH) models are in spirit very similar to their univariate counterparts. The main difference is in fact that mGARCH models specify equations for how the covariances move over time. Several different mGARCH formulations have been proposed in the academic literature, including VECH, diagonal VECH, DVEK, CCC-GARCH[6] and orthogonal GARCH models. Since the complexity of majority of models is emphasized, the deliberate consideration with respect to theoretical propositions and practical issues is needed.

VECH multivariate GARCH (1,1). A common specification of the VECH model according to [8] is given as:

$$\mathbf{r_t} = \boldsymbol{\mu}_\mathbf{t} + \boldsymbol{\varepsilon}_\mathbf{t} \quad (17)$$

$$\mathbf{V_t} = \mathbf{W} + \mathbf{A}\,\boldsymbol{\varepsilon}_{\mathbf{t-1}}\boldsymbol{\varepsilon}_{\mathbf{t-1}}^{\mathbf{T}} + \mathbf{B}\mathbf{V}_{\mathbf{t-1}} \quad (18)$$

where $\mu_t = [\mu_1,...,\mu_n]^T$ is the vector of mean returns and $\varepsilon_t = [\varepsilon_1,...,\varepsilon_n]^T$ is the vector of random shocks which conditional variance is represented by the *n*-by-*n* matrix $\mathbf{V_t}$. In the multivariate GARCH specification, the model parameters **A** and **B** are positive definite, *n*-by-*n* matrices and **W** is *n*-by-1 matrix. The art of building multivariate GARCH models is to specify the dependence of $\mathbf{V_t}$ on the past in such a way that $\mathbf{V_t}$ always remains symmetric and positive definite. The Equation (18) can be represented in a VECH operator form as:

$$VECH(\mathbf{V_t}) = \mathbf{W} + \mathbf{A}\,VECH\left(\boldsymbol{\varepsilon}_{\mathbf{t-1}}\boldsymbol{\varepsilon}_{\mathbf{t-1}}^{\mathbf{T}}\right) + \mathbf{B}\,VECH(\mathbf{V}_{\mathbf{t-1}}) \quad (19)$$

where the VECH operator takes the 'upper triangular' portfolio of a matrix and stacks each element into a vector with a single column [14]. For example, in the two asset case we have VECH ($\mathbf{V_t}$) = [$\sigma_{1,1}$, $\sigma_{1,2}$, $\sigma_{1,3}$]^T where $\sigma_{i,\,i,t}$ represents conditional variances at time *t* of the each asset in portfolio. The terms $\sigma_{i,\,j,t}$ for $i \neq j$ denotes the conditional, time dependent, covariances between the asset returns. The Equation (19) in the matrix form for the two variable becomes:

$$\begin{bmatrix} \sigma_{1,t}^2 \\ \sigma_{21,t} \\ \sigma_{2,t}^2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t+1}^2 \\ \varepsilon_{1,t=1}\varepsilon_{2,t-1} \\ \varepsilon_{2,t-1}^2 \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \begin{bmatrix} \sigma_{1,t-1}^2 \\ \sigma_{21,t-1} \\ \sigma_{2,t-1}^2 \end{bmatrix} \quad (20)$$

Equivalently,

$$\sigma_{1,t}^2 = \omega_1 + \alpha_{11}\varepsilon_{1,t-1}^2 + \alpha_{12}\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \alpha_{13}\varepsilon_{2,t-1}^2 + \beta_{11}\sigma_{1,t-1}^2 + \beta_{12}\sigma_{21,t-1} + \beta_{13}\sigma_{2,t-1}^2 \quad (21)$$

$$\sigma_{21,t} = \omega_2 + \alpha_{21}\varepsilon_{1,t-1}^2 + \alpha_{22}\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \alpha_{23}\varepsilon_{2,t-1}^2 + \beta_{21}\sigma_{1,t-1}^2 + \beta_{22}\sigma_{21,t-1} + \beta_{23}\sigma_{2,t-1}^2 \quad (22)$$

$$\sigma_{2,t}^2 = \omega_3 + \alpha_{31}\varepsilon_{1,t-1}^2 + \alpha_{32}\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \alpha_{33}\varepsilon_{2,t-1}^2 + \beta_{31}\sigma_{1,t-1}^2 + \beta_{32}\sigma_{21,t-1} + \beta_{33}\sigma_{2,t-1}^2 \quad (23)$$

It can be inferred that conditional variances and conditional covariances depend on the lagged values of all of the conditional variances of, and conditional covariances between, as well as the lagged squared shock values and the error cross-products [19].

A major problem with most multivariate GARCH specifications is that the number of parameters tends to explode with the dimension of the model, making them unsuitable for analyses of many risk factors. The number of parameters in VECH model is (N× (N + 1) + N^2 × (N + 1)^2) / 2. In the two variable case presented above the number of parameters of this model is 21. Furthermore, the specification of $\mathbf{V_t}$ is not guaranteed to be positive semi-definite. In practice it is therefore usually necessary to restrict the model both to contain the curse of dimensionality and ensure positive definiteness. Trying to estimate such a model is bound to be difficult, not only may it take a time to estimate parameters, but there are multiple local optima in the likelihood function used which requires the use of multiple different starting values. Thus, in general, a simplified version of the model, for the practical purposes, is used. [11]

Diagonal VECH multivariate GARCH (1,1). In the further VECH development, according to [8], the diagonal VECH (DVECH) has been suggested. The most common simplification has been to restrict attention to cases when matrices **A** and **B** of the Equation (19) are diagonal matrices. This special case can be written as:

$$VECH(\mathbf{V_t}) = \overline{\mathbf{W}} + \overline{\mathbf{A}}\,VECH\left(\boldsymbol{\varepsilon}_{\mathbf{t-1}}\boldsymbol{\varepsilon}_{\mathbf{t-1}}^{\mathbf{T}}\right) + \overline{\mathbf{B}}\,VECH(\mathbf{V}_{\mathbf{t-1}}) \quad (24)$$

where and the and must all be symmetric matrices such that has positive diagonal elements and all other matrices have non-negative diagonal elements. This reduces the number of parameters to be estimated to 3N × (N+1) / 2, or 9 parameters in the two asset and we have:

6 Constant Conditional Correlation model

$$\begin{bmatrix} \sigma^2_{1,t} \\ \sigma_{21,t} \\ \sigma^2_{2,t} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix} \begin{bmatrix} \varepsilon^2_{1,t+1} \\ \varepsilon_{1,t=1}\varepsilon_{2,t-1} \\ \varepsilon^2_{2,t-1} \end{bmatrix} + $$

$$+ \begin{bmatrix} \beta_{11} & 0 & 0 \\ 0 & \beta_{22} & 0 \\ 0 & 0 & \beta_{33} \end{bmatrix} \begin{bmatrix} \sigma^2_{1,t-1} \\ \sigma_{21,t-1} \\ \sigma^2_{2,t-1} \end{bmatrix} \qquad (25)$$

Equivalently,

$$\sigma^2_{1,t} = \omega_1 + \alpha_{11}\varepsilon^2_{1,t-1} + \beta_{11}\sigma^2_{1,t-1} \qquad (26)$$

$$\sigma_{21,t} = \omega_2 + \alpha_{22}\varepsilon_{1,t-1}\varepsilon_{2,t-1} + \beta_{22}\sigma_{21,t-1} \qquad (27)$$

$$\sigma^2_{2,t} = \omega_3 + \alpha_{33}\varepsilon^2_{2,t-1} + \beta_{33}\sigma^2_{2,t-1} \qquad (28)$$

By doing this, model implies that we return a uGARCH (1,1) model for each of the volatility terms, but there is also a covariance term that has to be estimated by maximum likelihood method underlying multivariate distribution. Especially, this may lead to time consuming process, when huge covariance matrices are imposed by number of positions in portfolio. Moreover, in some cases, the covariance matrix in DVECH model may be not positive definite. [19]

Constant Conditional Correlations multivariate GARCH (1,1). The convergence and estimation problems of time-varying covariances in multivariate GARCH models lead to the so called Constant Conditional Correlations multivariate GARCH (1,1) or CCC-mGARCH(1,1) model for practical purposes usage. According to [8] there is a possibility to retain the time varying properties by using conditional variances and keeping correlations constant through the time. Consequently, conditional covariance matrix is of the form:

$$\mathbf{V_t} = \mathbf{D_t R D}_t \qquad (29)$$

$$\mathbf{D_t} = \begin{bmatrix} \sqrt{\sigma^2_{1,t}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{\sigma^2_{k,t}} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,k} \\ \vdots & \ddots & \cdots & \vdots \\ \rho_{k-1,1} & \rho_{k-1,2} & 1 & \rho_{k-1,k} \\ \rho_{k,1} & \rho_{k,2} & \cdots & 1 \end{bmatrix} \qquad (30)$$

where **R** is constant, positive-definite correlation matrix and Dt is a diagonal volatility matrix with elements satisfying univariate GARCH (1,1) given by Equation (15). This constant conditional correlation specification represents a simple way of combining univariate GARCH processes with multivariate logic. By this, each asset volatility term follows univariate GARCH (1,1). As a result, this model has K(K+5)/2 parameters, moreover this specification guaranties positive definiteness and identification of $\mathbf{V_t}$.[11]

The CCC-mGARCH(1,1) model is often a useful starting point from which to proceed to more complex models. In some empirical settings it gives an adequate performance, but it is generally considered that the constancy of conditional correlation in this model is an unrealistic feature and that the impact of news on financial markets requires models that allow a dynamic evolution of conditional correlation as well as a dynamic evolution of volatilities.

Orthogonal GARCH (1,1). So called, "Ortogonal GARCH" (OGARCH) represents the solution for the problem of huge number of covariance matrix parameters and multivariate maximum likelihood estimation difficulties. For any application, the covariance matrix may be large and hence difficult to work with. The *principal components analysis* (PCA) is method for extracting the most important independent sources of information in the data. This approach is computationally efficient because it allows an enormous reduction in the dimensionality of the problem, whilst retaining a very high degree of accuracy. This also allows an enormous reduction in number of parameters that has to be estimated applying multivariate GARCH logic. By this idea, we can find and use linear combinations of principal components, which are, by definition uncorrelated, and by using only them we reduce problem dimensionality during parameter estimation process.

Let us denote by **r** the *T* observations of matrix which comprises of *n* assets. PCA will give up to *k* uncorrelated stationary variables, called the principal components, where each component is being a simple linear combination of the original returns. At the same time it is stated exactly how much of the total variation in the original system of risk factors is explained by each principal component. Each principal component is ordered according to the amount of variation it explains [1]. The results of PCA are sensitive to rescaling of the data and so it is standard practice to normalize the data before the analysis, for example assuming Normal distribution we subtract the sample mean and divide it by the sample standard deviation. If we define the diagonal matrix $\mathbf{V}=\{\sigma_1^2,\ldots,\sigma_n^2\}$ as a matrix of the empirical variances of vector $\mathbf{r}_t$ the standardized returns $\mathbf{z_t}$ are given as:

$$\mathbf{z_t} = \mathbf{V}^{-\frac{1}{2}}\mathbf{r_t}, \quad \mathbf{E}[\mathbf{z_t}] = 0, \quad \mathbf{E}[\mathbf{z_t z_t^T}] = \mathbf{V} \qquad (31)$$

then the **V** represents unconditional covariance matrix of $\mathbf{z}_t$ matrix . This matrix can be decomposed as:

$$\mathbf{V} = \mathbf{P\Lambda P^T} \tag{32}$$

where **P** is the orthonormal eigenvectors matrix which each column corresponds to the eigenvalue λ_i where $i = 1,..,n$. Matrix **Λ** is the diagonal matrix of the eigenvalues of **V** such that **Λ** = *diag*š { λ,…, λ_n }is ranked in descending order $\lambda_1 > \lambda_2 \ldots > \lambda_n$. Hence **V** can be written as:

$$\mathbf{V} = \mathbf{P\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}}P^T} = \mathbf{LL^T} \tag{33}$$

where $\mathbf{L} = \mathbf{P\Lambda^{\frac{1}{2}}}$ denotes *Cholesky* decomposed component of matrix **V.** Matrix **L** satisfies:

$$\mathbf{L^TL} = \mathbf{I_N} = \mathbf{LL^T} \tag{34}$$

According to [9], denote by $\mathbf{y_t}$ the vector of principal components of $\mathbf{z_t}$, which is defined by:

$$\mathbf{y_t} = \mathbf{L^{-1}z_t} \tag{35}$$

This expression can be interpreted as the return of a portfolio that assigns weight y_{ij} to the j-th security. Since principal components have the property that they are uncorrelated, this implies that during the modeling of the covariance matrix we can ignore the covariance terms and hence model the variance by each principal component separately. Consequently the problem reduces to a series of univariate estimations using GARCH (1,1) model. Another important property of this analysis is that the variance of each principal component is the corresponding eigenvalue. Note that:

$$E(\mathbf{y_t}) = \mathbf{L^{-1}}E(\mathbf{z_t}) = 0 \tag{36}$$

The unconditional covariance matrix of Equation (35) becomes:

$$E(\mathbf{y_t y_t^T}) = \mathbf{L^{-1}}E(\mathbf{z_t z_t^T})\mathbf{L^{-1T}} = \mathbf{L^{-1}VL^{-1T}} = \mathbf{I_n} \tag{37}$$

since Equation (33) holds, $\mathbf{y_t}$ is cross-sectionally uncorrelated and each component has a unit variance. Since each $\mathbf{z_t} = \mathbf{L\,y_t}$ coordinate of $\mathbf{z_t}$ can be written as a linear combination of the principal components:

$$z_{t,i} = \sum_{j=1}^{n} l_{i,j} y_{t,j} \tag{38}$$

where i=1,…n and $l_{i,j}$ and are the elements of **L**.

On the other hand, conditionally on the information available at *t-1*, the vector of standardized residuals $\mathbf{z_t}$ has a zero mean and a covariance matrix $\mathbf{V_t}$. That is:

$$E[\mathbf{z_t} \mid \Psi_{t-1}] = E[\mathbf{z_t}] = 0\,, \quad E[\mathbf{z_t z_t^T} \mid \Psi_{t-1}] = \mathbf{V_t} \tag{39}$$

where, for any t, the matrix $\mathbf{V_t}$ is positive definite and measurable with respect to the information set ψ_{t-1} so we have that:

$$\widetilde{\mathbf{V}}_\mathbf{t} = E[\mathbf{y_t y_t^T}] = \mathbf{L^{-1}V_tL^{-1T}} \tag{40}$$

Assuming that the conditional covariance $\mathbf{V_t}$ follows Equation (18) the multivariate VECH process, we can employ the orthonormal basis of principal components by applying the linear transformation Equation (35) to the conditional residuals $\mathbf{z_t}$. In the orthonormal basis of principal components Equation (18) then reads:

$$\widetilde{\mathbf{V}}_\mathbf{t} = \widetilde{\mathbf{W}} + \widetilde{\mathbf{A}}\,\widetilde{\mathbf{y}}_{\mathbf{t-1}}\widetilde{\mathbf{y}}_{\mathbf{t-1}}^{\mathbf{T}} + \widetilde{\mathbf{B}}\widetilde{\mathbf{V}}_{\mathbf{t-1}} \tag{41}$$

Where $\widetilde{\mathbf{y}}_{\mathbf{t-1}} = \mathbf{L^{-1}y_t}$ and $\widetilde{\mathbf{M}} = \mathbf{L^{-1}ML^{-1T}}$ for $\mathbf{M} \in \{\mathbf{V,A,B}\}$. and for . The purpose of stated equation is the ability to estimate separately each principal component of the conditional covariance matrix of principal components with respect to information set ψ_{t-1}. Since the principal components are orthogonal, it is reasonable to assume that the matrix $\widetilde{\mathbf{V}}$ is diagonal. In this case, the process given by Equation (18) can be estimated separately for each principal component which gives a set of n independent scalar equation of the GARCH (1,1) form.

Once we estimate the set of parameters in Equation (41) we can apply the inverse transformation:

$$\mathbf{V_t} = \mathbf{L}\widetilde{\mathbf{V}}_\mathbf{t}\mathbf{L^T} \tag{42}$$

to retrieve the series of conditional covariance matrices in the original basis of standardized returns. This allows us to estimate VaR in multivariate framework in most efficient way with loosing no information. [22]

The estimation of the elements of $\widetilde{\mathbf{V}}$ is computationally much simpler and faster than the original conditional variance-covariance matrix $\mathbf{V_t}$. The dimensionality of the problem is thus reduced to estimation of only n parameters.

Fitting Multivariate GARCH Models. In practice, the most widely used approach to fitting GARCH models to data is maximum likelihood estimation (MLE) method[7]. In this paper, we consider application of MLE method with Normal and Student t distribution as an underlying assumptions. Since fitting model parameters underlying multivariate distributions with higher dimensions, by applying MLE may not be feasible and it is not recommended, we combine apply estimation of multivariate GARCH models which may be treated as a set of univariate GARCH counterparts. In an ideal factor model we would have a diagonal covariance matrix. This means that we fit both CCC-mGARCH model where the constant conditional correlation matrix is the identity matrix and O-mGARCH model with its diagonalized form of principal components. The log-likelihood function underlying standard Student's-t distribution reduces to:

$$L_s = -\frac{1}{2}\sum_{t=1}^{T}\left[ln\sigma_t^2 + (\nu+1)ln\left(1 + \frac{r_t^2}{(\nu-1)\sigma_t^2}\right)\right] \tag{43}$$

and for Normal distribution it reduces to:

$$L_n = -\frac{1}{2}T \cdot ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\left[ln\sigma_t^2 + \frac{r_t^2}{\sigma_t^2}\right] \tag{44}$$

[7] **The maximization is performed by using a modified Newton–Raphson procedure**

4. Empirical study and results

This empirical study consists of applying VaR techniques described in previous chapters on a practical, real life financial data. The VaR estimate for each methodology is based on the same underlying hypothetical portfolio. The constructed hypothetical portfolio consists of stocks, treasuries and foreign exchange rates from the Serbian capital market that could be hypothetically held by any Serbian bank. The graphical representations of the VaR estimation results are presented and compared with daily profits/losses incurred by holding this hypothetical portfolio. The Basel II backtesting[8] procedure, together with Kupiec test, is used to decide which model is the most appropriate to use on day-to-day basis for assessing VaR and therefore determining the market risk capital requirement.

The required input data for this empirical study includes necessary information about financial instruments of the portfolio considered. Data on portfolio constituents such as: time history of prices, positions values, denomination currency for each portfolio position was necessary to form the whole portfolio and portfolio constituent's returns. All those data represents the main input for the VaR analysis. In order to evaluate the models, equally weighted hypothetical portfolio, that has been constructed and used in VaR estimation analysis, comprises of:

- ✓ 5 stocks, denominated in RSD, continuously traded on Belgrade Stock Exchange, namely: AGBN, AIKB, ENHL, PRBN and TIGR.
- ✓ 3 foreign exchange positions: EUR, USD and CHF position. The exchange rate for these foreign exchange currencies, with respect to RSD.
- ✓ 2 zero-coupon treasuries bonds continuously traded on Belgrade Stock Exchange: A2010, A2011. Each treasuries bond is EUR denominated. The first matures at 31. may 2010 and the second matures at 31. may 2011.

The price time series for those 10 portfolio positions are obtained from BELEX data feed stream (www.belex.rs). Data price time series ranges from 20. September 2007 to 11. September 2009. This data range has been chosen for the risk estimation purpose to emphasis the bullish period of the economy due to financial global crisis and to assess the VaR for this harsh period for the Serbian economy. In total we have 501 price observations for each portfolio position which are sorted in ascending order with respect to date. For the multivariate VaR estimation purposes each of 10 price time series is used and transformed in 500 corresponding return observations according to Equation (1). The day-to-day positions values are calculated assuming approximately equally weighted portfolio. This means that each of ten portfolio positions approximately captures 10% of total portfolio amount on the daily basis. Moreover, since the portfolio is constructed upon components denominated in different currencies, the arithmetic return is calculated after price time series of each portfolio components were transformed to be expressed in RSD currency. In other words, time series of daily prices were firstly calculated to be RSD denominated and then corresponding returns have been calculated.

Count	*Date*	*AGBN*	*AIKB*	*CHF*	*EUR*	*USD*	*ENHL*	*PRBN*	*TIGR*	*A2010*	*A2011*	*PORTFOLIO*
1	21/09/2007	0.313%	4.356%	0.013%	-0.223%	-1.016%	1.984%	1.079%	-1.598%	-0.488%	-0.223%	0.811%
2	24/09/2007	1.697%	0.399%	-0.721%	-0.588%	-0.687%	-2.645%	-0.194%	1.578%	-0.565%	-0.637%	-0.074%
3	25/09/2007	4.170%	-0.468%	-0.771%	-0.753%	-0.563%	1.499%	0.331%	-2.056%	-0.501%	-0.487%	0.342%
4	26/09/2007	0.292%	0.106%	-0.116%	-0.249%	-0.721%	1.077%	3.374%	2.099%	-0.249%	-0.249%	1.069%
5	27/09/2007	0.444%	0.053%	0.058%	0.325%	0.332%	-0.152%	1.332%	0.046%	-0.219%	0.044%	0.378%
6	28/09/2007	-1.915%	0.062%	0.168%	0.532%	0.390%	-0.915%	0.703%	-0.913%	0.812%	0.532%	-0.194%
7	01/10/2007	0.454%	0.000%	-0.119%	0.043%	-0.651%	-0.062%	-0.717%	0.691%	-0.188%	0.043%	-0.072%
8	02/10/2007	-0.458%	-0.991%	-0.267%	-0.237%	0.057%	-1.016%	0.352%	0.686%	-0.237%	-0.237%	-0.213%
9	03/10/2007	-0.078%	1.028%	-0.222%	-0.102%	0.243%	-0.093%	-0.369%	0.455%	0.813%	-0.102%	0.125%
10	04/10/2007	-0.203%	0.557%	-0.404%	-0.500%	0.043%	0.498%	-0.926%	1.357%	-0.832%	-0.573%	-0.055%
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
491	31/08/2009	-0.255%	5.646%	0.068%	-0.228%	0.296%	1.139%	0.000%	1.883%	-0.123%	0.433%	0.614%
492	01/09/2009	-3.809%	0.445%	-0.328%	-0.150%	-0.616%	3.041%	-1.460%	2.826%	0.340%	-0.238%	0.001%
493	02/09/2009	-2.135%	-1.693%	0.279%	0.186%	1.159%	-1.639%	-3.111%	0.529%	-0.637%	0.219%	-0.337%
494	03/09/2009	0.339%	0.082%	0.501%	0.309%	-0.113%	-1.222%	2.294%	0.000%	0.645%	0.309%	0.289%
495	04/09/2009	0.383%	2.786%	0.016%	0.062%	0.209%	2.475%	0.149%	1.472%	0.114%	0.029%	0.562%
496	07/09/2009	7.484%	4.544%	-0.275%	-0.097%	-0.648%	0.439%	0.000%	1.658%	-0.003%	-0.097%	0.864%
497	08/09/2009	1.629%	3.889%	-0.255%	-0.117%	-0.159%	2.732%	4.627%	0.815%	0.029%	-0.424%	0.757%
498	09/09/2009	-1.726%	-0.550%	0.254%	0.056%	-0.987%	0.957%	9.130%	1.112%	-0.048%	0.331%	0.471%
499	10/09/2009	1.160%	6.531%	0.106%	0.060%	-0.531%	0.738%	4.444%	-2.100%	0.133%	0.027%	0.585%
500	11/09/2009	1.447%	6.339%	0.230%	0.138%	-0.143%	3.766%	3.129%	0.919%	0.325%	0.138%	1.169%

Table 1: *Periods of high and low volatility*

[8] **Backtesting represents the routine of comparing daily profits and losses, the trading outcomes, with model generated VaR estimate to gauge the accuracy of it. Results, or in other words outputs, of backtesting are recognized as the number of exceptions i.e. VaR breaches. The exception is captured each time when the amount of loss in trading portfolio of the bank exceeds the estimated VaR for that day. Backtesting routine involves systematically comparing the history of VaR forecasts with their associated portfolio profits and losses**

The whole data range shown in the Table 1 is divided into two parts:

I. Initial estimation period

II. VaR estimation period.

Initial estimation period comprises of the first 250 returns, from 1st to 250th return i.e. from *20. September 2007* to *16. September 2008*. Consequently, the first VaR results, with different multivariate GARCH methods, are calculated underlying the data from the initial estimation period, since those returns are used as an initial input for the first estimate of VaR. Moreover, the equal weights of portfolio constituents used are imposed with respect to prices of positions on the last day, *16. September 2008,* of initial estimation period. The portfolio returns and summary statistic are given in the following Table 2. The table below provides summary statistics as well as the Jarque-Bera statistic for testing normality. For all portfolio assets including portfolio itself, the null hypothesis of normality is rejected at any level of significance, as there is evidence of significant kurtosis and negative skewness. From Table 2, relatively low values of degrees of freedom (DoF) for all portfolio components including portfolio itself are provided, which confirms that there is a relatively high kurtosis observed in data.

	AGBN	*AIKB*	*CHF*	*EUR*	*USD*	*ENHL*	*PRBN*	*TIGR*	*A2010*	*A2011*	*PORTFOLIO*
Position*	100	300	26,300	16,600	23,600	900	1,000	1,300	18,600	19,700	12,945,751
Mean	-0.0034	-0.0030	0.0000	-0.0002	-0.0002	-0.0030	-0.0049	-0.0026	0.0000	-0.0001	**-0.0024**
Median	-0.0023	-0.0019	-0.0009	-0.0003	-0.0004	-0.0028	-0.0020	0.0000	-0.0005	-0.0006	-0.0025
St. Deviation	0.0262	0.0230	0.0075	0.0065	0.0086	0.0295	0.0338	0.0327	0.0070	0.0072	**0.0126**
Kurtosis	7.0192	11.3229	5.3011	7.9607	4.6473	4.6927	8.0507	4.9855	6.2435	6.8521	9.8930
Skewness	0.4425	0.2308	0.3045	0.2447	-0.0529	0.3920	1.0467	0.1347	0.2762	0.4914	0.8791
Minimum	-8.69%	-12.91%	-2.84%	-2.83%	-3.11%	-9.24%	-11.18%	-9.71%	-2.83%	-2.83%	-3.91%
Maximum	12.70%	12.79%	2.94%	2.96%	2.53%	10.02%	18.18%	10.00%	2.96%	2.96%	7.76%
Jarque-Bera stat.	176.43	723.7	59.02	258.8	28.38	36.25	311.36	41.82	112.76	164.63	527.13
p-value	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
Student's-*t* DoF	2.356	2.074	4.142	5.210	3.835	3.137	2.248	7.022	2.588	2.729	**3.454**
Count	250	250	250	250	250	250	250	250	250	250	250

*Stocks: indicates the number of stocks held. Treasuries: indicates the face value in foreign currency. Foreign exchange: indicates position in foreign currency.
Portfolio: position is expressed in RSD. Evaluation date: 16.september 2008.

Table 2: Summary statistics of the initial estimation period

VaR estimation period comprises of the second part of 250 returns in Table 3 ranging from 251st to 500th return i.e. from *17. September 2008* up to *11. September 2009*. Consequently, this is the observation period where VaR estimates are iteratively calculated on day to day basis. The VaR estimation, through this period, is done by applying "rolling window" concept. This means that the first VaR results are calculated underlying the data from the initial estimation period from 1st to 250th observation; then, the 1st observation (from the initial estimation period) is dropped out and the 252nd is included in "rolling window". Consequently, for the second VaR estimate we deal with range which captures returns from 2nd to 252nd and VaR is recalculated for each considered methodology according to newly imposed range. This procedure repeats iteratively until we estimate the VaR for last "rolling window" ranging from 251st to 500th return observation. The "rolling window" concept enables to deal with last 250 return observations in each period for which VaR is calculated. In this paper, there are 250 VaR estimates for 250 imposed rolling windows in range that denotes VaR estimation period.

Table 3. provides summary statistics for the VaR estimation period calculated on last day in observation period. The Table 3 shows that the average daily return is about zero percent, or at least negligibly small compared to the sample standard deviation. This is why the mean is often set at zero when modeling daily assets and portfolio returns. The Jarque-Bera statistic for testing normality indicates that in this period there is only one portfolio component ENHL whose returns followed Normal distribution.

For all other portfolio assets including portfolio itself, the null hypothesis of normality is rejected at any level of significance. Moreover, there is again an evidence of significant kurtosis and negative skewness. Again, from relatively low values of DoF is observed which confirms that there is a relatively high kurtosis observed in data. Furthermore, the maximum and minimum statistics are quite large in absolute value indicating the presence of extreme returns.

	AGBN	*AIKB*	*CHF*	*EUR*	*USD*	*ENHL*	*PRBN*	*TIGR*	*A2010*	*A2011*	*PORTFOLIO*
Position*	100	300	26,300	16,600	23,600	900	1,000	1,300	18,600	19,700	12,935,346
Mean	0.0000	-0.0006	0.0011	0.0008	0.0008	-0.0005	-0.0008	0.0004	0.0012	0.0012	**0.0001**
Median	0.0000	-0.0005	0.0007	0.0004	-0.0001	-0.0021	0.0000	0.0000	0.0011	0.0008	0.0007
St. deviation	0.0444	0.0478	0.0095	0.0067	0.0127	0.0404	0.0458	0.0299	0.0082	0.0076	**0.0103**
Kurtosis	5.4526	4.9877	8.5446	8.0631	4.2339	3.1247	4.6752	4.2895	10.2635	6.9532	7.5034
Skewness	0.1811	0.0782	-0.6077	-0.5650	0.2012	-0.0071	0.2818	0.2881	-0.2036	-0.3024	0.0866
Minimum	-16.43%	-18.91%	-5.32%	-3.60%	-3.94%	-10.02%	-13.28%	-9.00%	-4.22%	-3.83%	-4.78%
Maximum	18.75%	19.33%	3.49%	2.34%	4.80%	10.06%	17.14%	10.13%	4.59%	2.54%	5.18%
Jarque-Bera	64.02	41.41	335.6	280.32	17.54	**0.16141**	32.5408	20.778	551.3	166.6	211.56
p-value	0.001	0.001	0.001	0.001	0.0033	**0.500**	0.001	0.002	0.001	0.001	0.001
Student's-*t* DoF	2.7164	4.2697	2.3693	2.1119	5.6813	22.4680	3.1941	2.6883	2.4930	2.9426	**3.9368**
Count	250	250	250	250	250	250	250	250	250	250	250

*Stocks: indicates the number of stocks held. Treasuries: indicates the face value in foreign currency. Foreign exchange: indicates position in foreign currency. Portfolio: position is expressed in RSD. Evaluation date: 11.september 2009.

Table 3: *Summary statistics of the VaR estimation*

It is important to note that for each of 250 "rolling windows" summary statistics, DoF for Student-*t* distribution, together with all required VaR parameters such are recalculated to include return innovation effect from each "rolling window" into VaR estimates.

Empirical results are given for multivariate GARCH VaR techniques that has been discussed in previously chaptersa and which is possible to implement in practice. VaR measure has been calculated with respect to 99% confidence level and one day holding period. The tables below provide a graphical insight of calculated VaR methods.

Each of these methods has been implemented and fully programmed by authors in MATLAB version R2009a. The only toolbox used was Optimization toolbox for MLE maximization purposes. All other functions have been fully developed by the authors.

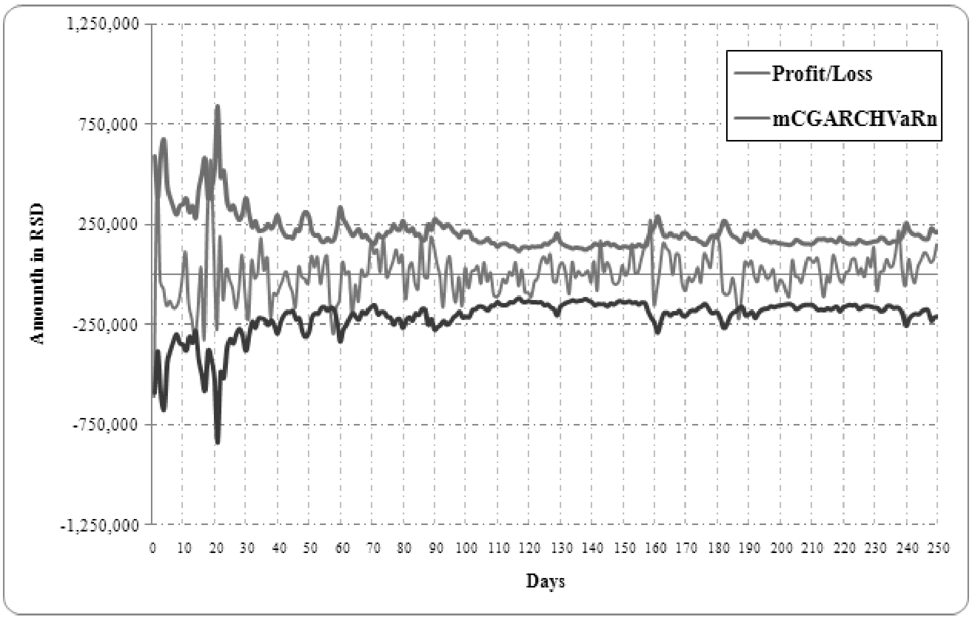

Figure 5: ***mCGARCHVaRn*** *– multivariate constant correlation GARCH (1,1) assuming normally distributed returns*

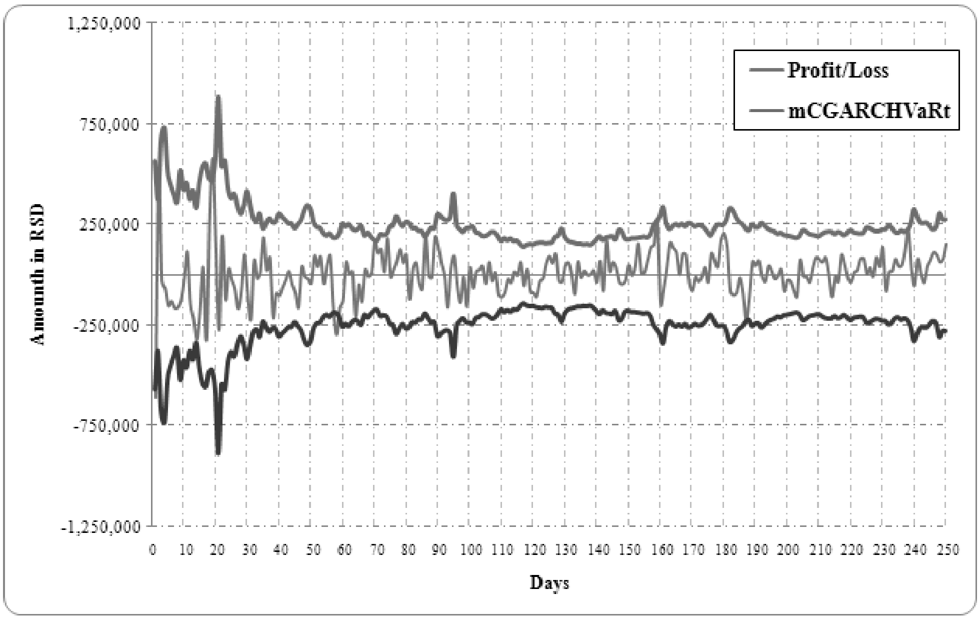

*Figure 6: **mCGARCHVaRt** – multivariate constant correlation GARCH (1,1) assuming Student's-t distributed returns*

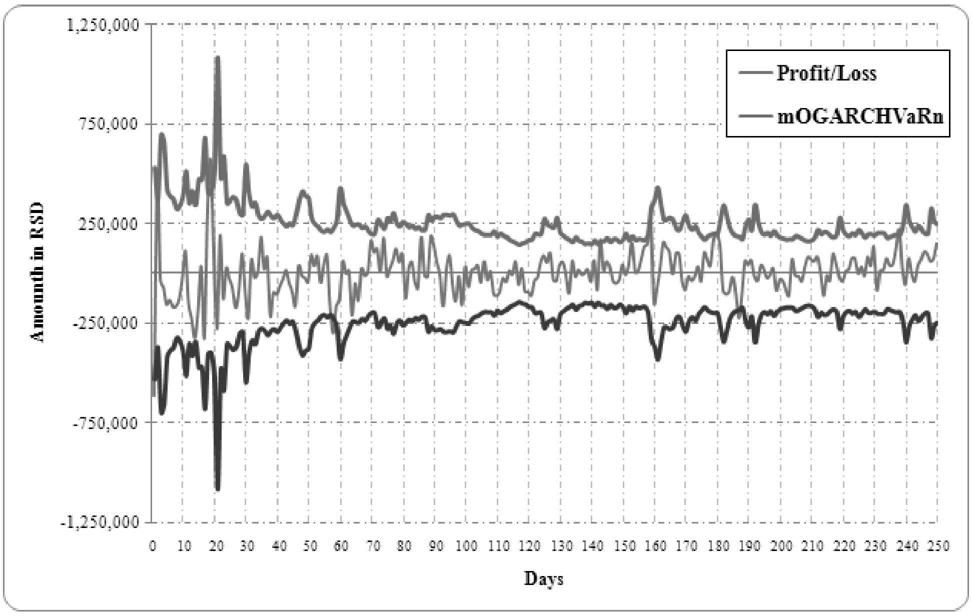

*Figure 7: **mOGARCHVaRn** – multivariate constant correlation GARCH (1,1) assuming Student's-t distributed returns*

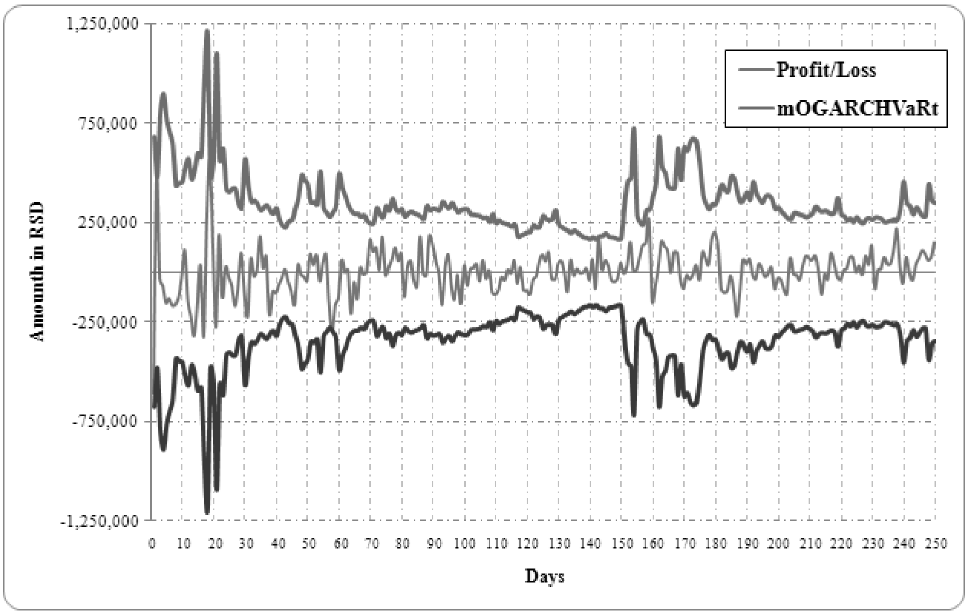

Figure 8: ***mOGARCHVaRt*** *– multivariate constant correlation GARCH (1,1) assuming Student's-t distributed returns*

The graphical results are shown above depict the evaluated VaR models together with the incurred profits/losses of the hypothetical portfolio. As it was discussed in previous chapters, VaR for the day *t* is compared with profits/losses for the day *t +1*. VaR estimates are shown as a two envelope red lines which stand on the both, profit and losses side of each figure. Breach is seized each time when blue line representing profit/loss, breaks through lower VaR envelope on the loss side.

VaR method	VaR exceptions					
	day 1	day 14	day 57	day 58	day 64	day 187
mCGARCHVaRn	-22,817	-43,833	-2,573	-127,342	-18,927	-59,534
mCGARCHVaRt	-45,227	0	0	-115,236	0	0
mOGARCHVaRn	-80,488	0	0	-71,010	0	-41,707
mOGARCHVaRt	0	0	0	-3,053	0	0

Table 4: *VaR breaches in the empirical analysis*

The number of exceptions with respect to 99% VaR ranges from 0 to 6 out of 250 observations, which is approximately close to expected number $250 \cdot 0.01 = 2.5$, imposed by confidence level. At the 1st day and 58th day of VaR calculation, the breaches are reached in almost each multivariate GARCH VaR technique. In the period ranging from 16. December 2008 to 16. June 2009 no exceptions were recorded. Consequently, it can be noticed that exceptions with respect to presented VaR models occur frequently around the same dates, which raises a doubt that there could have been some external market shocks. However, to get a better insight into the performance of the risk models, we proceed with formal statistical testing and interpretation of the obtained VaR results.

VaR method	Number of exceptions (NoE)	Basel II zone	Kupiec (accept for p > 0.05)	Average VaR	Average breach
mCGARCHVaRn	6	YELLOW	ACCEPT (0.0593)	212,597	45,838
mCGARCHVaRt	2	GREEN	ACCEPT (0.7419)	255,119	80,232
mOGARCHVaRn	3	GREEN	ACCEPT (0.7579)	255,650	64,402
mOGARCHVaRt	1	GREEN	ACCEPT (0.2780)	351,057	3,053

Table 5: *VaR backtesting analysis*

Table 5 shows the classification of VaR models according to the Basel II three-zone approach as required by the Basel II standards. From the Table 5 we also see that the minimum average VaR is calculated for mCGARCHVARn method and minimum average breach magnitude for mOGARCHVaRt method. The three out of four examined models are in green zone, but one VaR model is qualified to fall in yellow zone. This indicates that additional examination is needed in order to reveal potential the problems with their risk assessment. In order to test whether the occurrence of exceptions covered by VaR is in line with its confidence level and whether the losses occur independently of each other author applies Kupiec test. The decision making rules concerning the acceptance or rejection of the null hypotheses are based on the corresponding likelihood ratio test statistics and a significance level of 5%. In Table 5 the *p*–values, shown in brackets, are the probabilities which indicate failure rates significantly different from probability of one percent, at 95% Kupiec test level. Rejections arise if the frequency of violations produces $p<0.05$.

According to Basel II, since VaR results have been statistically validated it should be used for determining the minimum regulatory capital against market risks. The bank must meet, on a daily basis, a *market risk capital requirement* expressed as the higher of:

a) its previous day 10-day VaR number measured according to the specified parameters
b) an average of the daily 10-day VaRs measures on each of the preceding sixty trading days multiplied by a multiplication factor ($k+p$). Where k is usually set to 3, whereas p stands for potential increase in multiplication factor due to poor backtesting results[9].

For the capital requirements purposes "square root of time" rule is used to approximate 10-day VaR from the obtained 1-day VaR estimate.

$$\mathrm{VaR}_{\alpha,10\mathrm{day}} = \sqrt{\mathrm{T}} \cdot \mathrm{VaR}_{\alpha,1\mathrm{day}} \tag{45}$$

Day	Date	Capital requirement for market risk			
		mCGARCHVaRn	mCGARCHVaRt	mOGARCHVaRn	mOGARCHVaRt
1	17/09/2008	5,589,355	5,376,750	5,042,241	6,424,427
2	18/09/2008	4,606,742	4,479,332	4,290,507	5,490,305
3	19/09/2008	4,986,717	5,149,385	5,061,876	6,217,878
4	22/09/2008	5,329,657	5,601,625	5,360,787	6,785,079
5	23/09/2008	5,093,278	5,459,442	5,088,295	6,884,537
...	...	...	...	...	...
246	04/09/2009	549,597	715,950	625,292	887,451
247	07/09/2009	564,750	752,481	633,164	884,895
248	08/09/2009	726,373	967,517	1,031,379	1,394,188
249	09/09/2009	671,286	879,224	832,200	1,138,735
250	10/09/2009	659,041	874,804	780,262	1,091,479

Table 6: *Capital requirements for market risk*

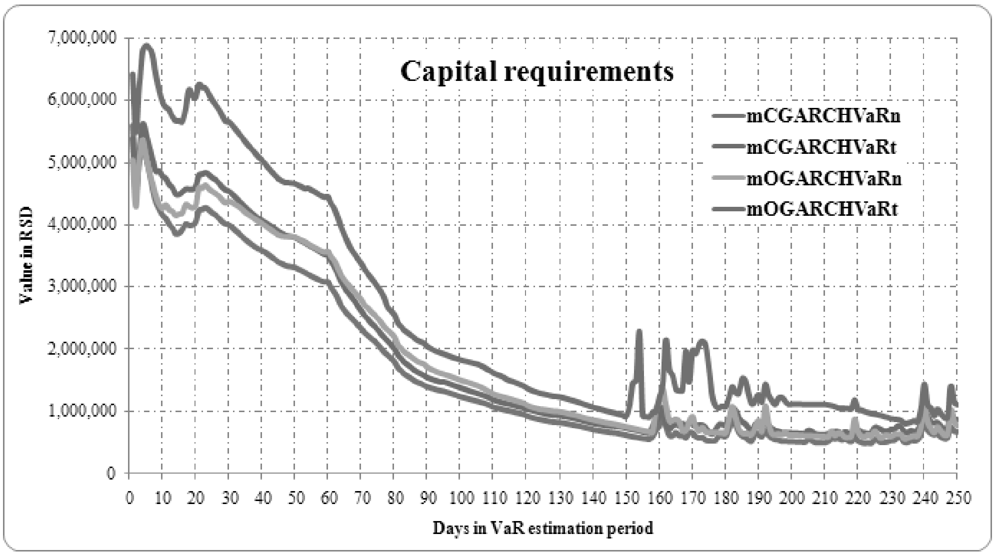

Figure 5: *Capital requirements comparison*

[9] For more details see [3]

The capital requirements time expansion is gathered on the Figure 5. The comparison shows, that within first 60 days, there are a higher capital requirements revealed then in the period after 60th day. The period after 60th day indicates stable decrease in capital requirements for all four methods. The mCGARCHVaRt keeps its volatile nature and there are a few outliers captured around 152nd, 170th, 185th and 240th day. By inspection mCGARCHVaRn has the lowest capital requirement for market risk. Its time line is stable up to the last VaR estimation day which indicates that some reasonable theoretically imposed limit on the market risk capital requirement would not be exceeded. On the other hand, there is an evidence, for all four methods, of a slight upward increase of the capital requirement trend lines which would perhaps indicate commencement of another volatile period for hypothetical portfolio profits and losses.

Conclusion

Value-at-Risk model evaluation represents a crucial part of market risk management practice. Its recognition and practical implementation is mainly motivated by the wide adoption of regulatory standards proposed by the Basel Committee on Banking Supervision. Theoretically, risk managers have abundance of which Value-at-Risk methodology to choose but there are a plenty of peculiarities that have to be deal with during this decision making process. For example, from the perspective which VaR measure is the most relevant, there are different criteria that have to be satisfied such as model validation, regulatory compliance and internal bank's standards. Therefore it is necessary to test and compare VaR estimates on the actual portfolio and to indicate its validity. In Serbia, for example, there is an emphasized need for choosing the appropriate VaR model due to convergence and compliance with Basel II standards.

In this paper a set of multivariate GARCH models, which represent advanced quantitative VaR estimation techniques, is discussed, empirically evaluated and tested. Furthermore, Normal and Student's *-t* distributional assumptions are met and investigated within those VaR techniques. Finally, the whole set of methods is used and examined in order to find the most appropriate VaR models for 99% confidence level and 1 day holding period. Regulatory recommended backtesting procedures are used in order to validate the considered VaR models and to choose the most appropriate. Two approaches to backtesting are followed since the validation of results has direct implications on decision making process concerning election of adequate VaR method within a bank, as well as, on the level of capital requirements for market risk. Basel II "three zone" test has been applied as well as the Kupiec tests based on the frequency of tail losses. The global objective of this paper was to determine and improve the accuracy and adequacy of risk modeling in emerging market, such Serbia is, for the practical banking purposes concerning market risk capital requirements calculations.

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